Hedging involves eliminating risk in one investment by taking an offsetting position in another investment. For example, shorting index futures contracts with equivalent notional principal to an underlying index investment or buying put options with a strike close to the current market price in a quantity matching the portfolio will each hedge against a decline. In both of these cases, matching the hedge size to the portfolio size minimizes the need for adjustments. Dynamic hedging by comparison refers to a hedging strategy that requires periodic adjustment. The benefit of this additional effort is that the hedge can be established and maintained with a lower cost, with less upside compromised, or with both.

The seller of an options contract faces the risk of a loss from exercise of the option. Several strategies can mitigate this risk. The first is to buy exactly the number of underlying shares—the covered strategy. Suppose that one vanilla call is sold; the writer buys 100 shares. If the call expires in the money, the writer sells the shares at a profit, but hands over the profit to the buyer. This strategy is not optimal. First, there is the opportunity cost of tying up cash to buy 100 shares. Second, if the call expires out of the money, the writer will be freed of his obligation on the call but will hold shares that have depreciated in value. Finally, if the call expires in the money, a better strategy for the writer would have been to not write the call but to buy the shares as an investment. The second strategy to mitigate risk is to buy 100 shares as soon as the call goes in the money and sell the shares immediately when the option goes out of the money, and to do this continually over the life of the call. This stop-loss strategy could add substantial churn and generate small but regular losses. The better strategy is to delta hedge the call so that only the optimal number of shares are held at every instant of the call’s life. This is possible because the ratio of the price change of the call to the underlying, also known as the hedge ratio, is less than one. Delta, as the hedge ratio is properly called, defines the amount of the underlying security to be held during the call’s life in order for the writer’s portfolio (sold call plus bought shares) to be insensitive to changes in the price of the underlying.

Delta is not fixed but changes in response to changes in the underlying price. This change is measured by gamma. When delta changes rapidly, gamma is high and the portfolio must be rebalanced more often for it to remain riskless, which can lead to high transaction costs. The typical remedy is to gamma hedge the portfolio so that it is insensitive to changes in delta. Gamma hedging eliminates the constant adjustments required by pure delta hedging. Proper risk management of options therefore depends on how well delta, gamma, and other option price sensitivities (Greeks) are calculated. For vanilla options, delta and gamma are usually straightforward to calculate and well behaved. In this article we show that for barrier options, delta and gamma can be very poorly behaved, which makes dynamic hedging of these options both impractical and costly.

Barrier options offer another twist on the cost/benefit of more complex instruments. Barrier options are similar to vanilla options, except that a trigger is specified so that the option can either cease to exist (a knock-out barrier) or can become alive (a knock-in barrier) when the stock price crosses the barrier. A barrier option that is knocked out becomes void. Thus, a trader writing barrier call options can potentially up volume by writing contracts more frequently.

One appeal of barrier options is their ability to more finely match a targeted strategy. Vanilla options are useful when investors anticipate either upward or downward future movements in the stock price, but
barrier options are preferable when these anticipations are more detailed. An investor who is bullish overall can buy a vanilla call, but one who is bullish and expects a floor on prices can buy a down-and-out call. Barrier options do not force investors to pay for price scenarios that they believe to be improbable. Moreover, barrier options are usually cheaper than their vanilla counterparts.

There is a cost beyond the lesser premium obtained from selling knock-out calls as compared to standard calls. Barrier options are much more difficult to hedge. In particular, delta from reverse barrier options can quickly become confusing when the stock price approaches the barrier. This makes the risk management of reverse barrier options difficult.

In this article we consider regular and reverse single barrier options only, but barrier options can have many features, including rebates, double barriers, soft barriers, barriers that reset, and they can come in European or American varieties.

DELTA

The delta of a vanilla European call at expiry is either zero or one. If the call expires out of the money and worthless, then delta is zero since any change in the stock price (provided it remains below the strike) will not have an effect on the option value, which will remain zero regardless. On the other hand, if the call expires in the money, then the call is worth its intrinsic value. Any $1 dollar increase in the stock price is matched exactly by a $1 increase in the call value, so the delta is one. This is illustrated by the dotted line in Exhibit 1.

With three months left until expiry, if the call is deep in the money, then delta is slightly less than one. This is because with three months left, there is a chance that the stock price will drop below the strike and render the call out of the money. Hence, a $1 dollar increase in the stock price will not be matched exactly by a $1 increase in the call value—the increase in value will be smaller. This implies that delta will be smaller than one. When the stock price is very high, it is unlikely that the stock price will drop below the strike price. Hence, delta eventually approaches one as the option becomes deeper in the money. Conversely, if the call is out of the money, then delta is slightly above zero, since with there is a chance that the stock price will rise above the strike price and render the call in the money. When the stock price is very low, this becomes unlikely, so delta approaches zero as the option becomes deeper out of the money. With one year left until expiry the argument is the same, except that large price movements in either direction are more likely. Hence, delta for the call with one year remaining is even further away from the limiting values of zero and one.

BARRIER OPTIONS

Regular knock-out options resemble their plain vanilla counterparts in both payoff and value. This is because these options cease to exist when the option is out of the money, and consequently, when the option has little value. Regular knock-in options also resemble vanilla options, since these come into existence when the option is in the money and has high value.

Exhibit 2 plots the value of a down-and-out (DO) call with barrier $90 and strike $100, with three months to expiry, and the value of a European vanilla call with the same features. We use formulas for barrier options presented in Haug [2007]. A DO call is a regular knock-out option. When the stock price is above the strike of $100 the DO call and vanilla call are both in the money and their values are almost identical. When the stock price falls below $100 both options fall out of the money and have only time value. When the stock price falls below the barrier of $90, however, the DO call gets knocked out and immediately loses all its value, but the vanilla call still has some time value left. As we move further to the right and deeper in the money, the DO call price converges to that of the plain vanilla call, since in that region a large downward move in the stock price becomes increasingly less likely, and consequently, so does the likelihood of the barrier being breached.

DELTA FOR REGULAR BARRIER OPTIONS
Given the preceding argument, we expect the delta of a deep in-the-money DO call to resemble the delta of a deep in-the-money vanilla call, but that the similarity would break down as we move out of the money. This is illustrated in Exhibit 3 for DO calls with barrier at $75, and for vanilla calls.

The behavior of delta within the in-the-money region above the strike of $100 is vastly different from the out-of-the-money region. In particular, above the strike the deltas of the DO calls match the delta of vanilla calls closely—almost perfectly for the three-month calls. Below $100, however, the relationship starts to break down, especially for the one-year calls. In particular, when the barrier of $75 is breached, the DO calls get knocked out, which produces a discontinuity in the DO deltas. It is also evident that the seller of a DO call needs to hold more stock to hedge the position than the seller of a vanilla call, since delta is higher for the DO call, especially for the one-year call. At the strike price of $100, the one-year vanilla call has a delta of 0.59, but the DO call has a delta of 0.68. As the stock price approaches the barrier, it becomes more and more likely the seller of the DO call will be freed of his obligation. The penalty is a higher hedging cost, as indicated by an increased delta: The seller has to hold more stock than if a plain vanilla call had been sold. Furthermore, the DO calls were sold at a lower premium. A similar diagram can be drawn for down-and-in puts, which are also regular barrier options (see Derman et al. [1996]). A trader undertaking such a strategy is dependent on a knock-out rate sufficiently high to make up the lost income with higher turnover.

**DELTA FOR REVERSE BARRIER OPTIONS**

Contrary to regular barrier options, which have a barrier that is out of the money, reverse barrier options have a barrier that is in the money. As a result, the delta from these options is often even more erratic. In particular, reverse barrier calls can have deltas that take on negative values, and reverse barrier puts can have deltas that take on positive values.

Exhibit 4 plots the value of an up-and-out (UO) call option with various times to expiry along with a vanilla call option with identical features and with one week to expiry. Consider the UO call with one week to expiry and the vanilla call with one week to expiry. For stock prices below $112, the value of both options is almost identical and are increasing with the stock price—in other words, their delta is positive. When the stock price rises above $118, the UO call starts to lose value, even though it is deep in the money. This is because of the effect of the barrier. As the stock price approaches the barrier, it becomes more and more likely that the barrier will be breached, and consequently, that the option will become worthless. As explained by Taleb [1997], the effect of moneyness and the barrier are two forces acting in opposite directions on the option’s value—moneyness increases value, but the barrier decreases value. Moreover, since this effect is acting in regions where the call option has a high value, the values of the UO and vanilla calls are highly divergent. Note also that delta for the UO call starts to become negative at $118. Again, this is due to the barrier. Unlike the vanilla call, past $118 an increase in the stock price is accompanied by a decrease in the value of the UO call; its delta is therefore negative.

For UO calls with longer expiries, the argument is the same. The value of these longer-dated barrier calls, however, starts to decrease at lower stock prices. The UO call with one-month expiry, for example, starts to lose value at around $113. This is because when the stock price is $113 there is a greater chance that the barrier will be breached in one month than in one week. This is reflected in a decrease in value of the one-month up-and-out call at $113; the value of one-week UO call at $113 is hardly affected. At $102, the three-month UO call becomes cheaper than the one-month UO call. This is because at that price the model starts to attach a greater likelihood of the barrier being breached in three months than in one month. This is contrary to vanilla calls. Indeed, longer-dated vanilla calls are always more expensive than shorter-dated calls, regardless of the stock price.

Finally, if the stock volatility were lower, the value of the UO calls would lie closer to the vanilla values in regions below the barrier, especially for shorter-dated UO calls. This is because a lower volatility decreases the chance of large movements in the stock price. Only when the stock price starts to approach the barrier very closely will the value of the UO calls start to drop.

We saw in Exhibit 4 that the delta of UO will become negative as the stock price approaches the barrier
and as the negative effect of the barrier starts to dominate the positive effect of moneyness. Exhibit 5 presents a plot of the delta of an UO call with one month, three months, and six months to expiry. The strike is $100, the barrier is set at $125, and the annual volatility is 25%. The deltas behave like their vanilla counterparts only when the stock price is very low and the calls are out of the money. All the deltas eventually become negative, and all suddenly become zero at the barrier. It is difficult to hedge UO calls using delta for these options, especially when they are in the money.

GAMMA

Gamma represents how fast delta changes in response to changes in the underlying stock price. When delta changes rapidly gamma is high and the hedge must be rebalanced more frequently. As noted earlier, gamma hedging is a remedy for rapidly changing delta. In the same way that a delta hedged portfolio has no exposure to changes in the stock price, a gamma hedged portfolio has no exposure to changes in delta. A gamma hedged portfolio will need less frequent rebalancing than a delta hedged portfolio, because gamma changes less dramatically than delta. Unfortunately, because of the unreliable behavior of their gammas, gamma hedging is simply not practical for barrier options.

To illustrate, Exhibit 6 plots gamma for a one month UO call and gamma for a vanilla call with identical features. For the vanilla call, gamma is near zero at the extremities of the horizontal axis, where delta changes very slowly and where the hedge need not be rebalanced frequently. It reaches its maximum at around $100 where delta is the steepest and where the most rebalancing must occur. The UO call gamma behaves identically to the vanilla gamma below $95. The large negative gamma between $105 and $125 corresponds to regions where the one-month delta in Exhibit 5 changes rapidly. Hedging in this region is costly because delta changes rapidly and difficult because delta changes signs. Gamma is infinite at the barrier, but this is not shown in the graph.

STATIC HEDGING

The erratic behavior of delta and gamma for barrier options can render dynamic hedging ineffective, costly, and in some cases near-impossible as a practical matter. Recognizing this, a number of researchers have proposed static and semi-static strategies as alternatives to dynamic hedging. The idea of these strategies is to create a portfolio that mimics the barrier option payoff, but that is made up of only vanilla options, and for this portfolio to be static in the sense that it need not be updated during the life of the barrier option. The portfolio of vanilla options must replicate the behavior of the barrier option as closely as possible.

The calendar-spread static hedging developed by Derman et al. [1995] uses a basket of vanilla options with different maturities to create a replicating portfolio that mimics the behavior of a UO call. If the barrier is never breached, the value of the portfolio is that of a vanilla call and the payoff is the intrinsic value of the call; if the barrier is breached before expiry, the value of the portfolio is zero. Hence, if we plot the value of the UO call when the stock price is at the barrier, its value should be zero during the entire life of the option, and then it should jump to its intrinsic value at expiry. The shape of this graph should therefore be a mirrored “L” shape. In reality, it will be only an approximation of this shape, as indicated in Exhibit 7. The approximation gets progressively more accurate as more and more vanilla calls are added to the replicating portfolio. An infinite number of vanilla calls would bring the approximation error to zero and create a perfect hedge.

Exhibit 7 illustrates and reproduces the analysis in Derman et al. [1995], using an UO call with strike at $100, barrier at $120, with one year until expiry, and when the volatility is 20%. The three replicating portfolio are each composed of a long position in a vanilla call with the same strike and maturity as the barrier option, along with long and short positions in a set of vanilla calls of varying maturities, but each with strike equal to the value of the barrier. When the stock price hits the barrier, the replicating portfolio has zero value, but it must be liquidated to avoid future liabilities on the vanilla calls that were sold to form the portfolio.
The plots in Exhibit 7 represent the value of replicating portfolios as the UO call approaches expiry, when the stock price is $120 (at the barrier). Suppose that we are two months into the life of the UO call (eight months remaining). At all times before that, the barrier must not have been hit, since the UO call is still alive—the barrier does get hit, however, at three months, so the value of the portfolio at that time is zero. Continuing this logic along the maturity axis towards expiry implies that the portfolio value is zero along all maturities, except at expiry. If the UO call is still alive at expiry, the barrier has not been breached. The value of the UO call at expiry is that of the vanilla call—the intrinsic value defined as the stock value of $120 minus strike of $100, or $20. Hence, the behavior is shaped like a reverse “L” that the portfolios attempt to replicate.

The replicating portfolio seeks to mimic the behavior of the UO call only at the boundary—namely at the barrier and at expiry. It seems incomplete that such a replication would also mimic the behavior of the UO call everywhere else. As pointed out by Dupont [2001], however, the value of a derivative is determined by its payoff at the boundary, so the strategy also guarantees replication within the boundary.

The approach of Derman et al. [1995] is only valid when volatility is constant over the life of the option. The hedge portfolio quickly loses its replicating accuracy when volatility is not constant, and it must be readjusted. Static hedging using put call symmetry or using optimal fitting are alternatives. These strategies are explained by Maruhn [2009].

CONCLUSION

To reduce the risk from writing vanilla options, delta and gamma hedges are set up: When the vanilla option is sold, a replicating portfolio is created that matches the payoff of the option, and this portfolio is rebalanced periodically. The effectiveness of these hedges requires that the option sensitivities (Greeks) be well behaved.

Barrier options are popular because they can better match investors’ beliefs about future price movements than can vanilla options, and they are usually cheaper. Market makers have welcomed the growth of OTC derivatives markets in general and of barrier options in particular, but have been faced with ineffective delta and gamma hedges. The Greeks of barrier options can experience dramatic swings, and can take on values that would be considered nonsensical for their vanilla counterparts. The erratic behavior of barrier option Greeks arises from the discontinuity in the option payoff.

The difficulty of hedging barrier options has prompted researchers to design static and semi-static hedging strategies that are designed specifically for barrier options. These strategies do not require rebalancing, so liquidity and transaction costs associated with dynamic hedging are much less of a consideration.

REFERENCES


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